Equivalent Models for Multi-terminal Channels

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***Abstract*—The recently introduced network equivalence results are used to create bit-pipe models that can replace multi-terminal channels within a discrete memoryless network. The goal is to create a set of simple "components" or "blocks" that can be substituted for the channel in such a way that the resulting network is capable of emulating the operation of the original one. We develop general upper and lower bounding models for the multiple access channel and for a class of broadcast channels. These bounds are sharp in the sense that there exists networks where the original channel can achieve the maximum sum rate permissible through the upper or lower bounding models. This approach provides a simple method for analyzing the capacity of large networks, which we illustrate with an example.**

I. Introduction

While a full characterization of the fundamental limits of general communication networks is still out of reach, several tools have been recently introduced to approach this problem (e.g., [1]–[3]). Nevertheless, the set of achievable rates and the optimal operation strategy of even simple networks, such as the three node relay channel, are unknown except for specific cases [3], [4].

In [3]–[5], Koetter *et al.* introduced the concept of *network equivalence*. Two networks are said to be "equivalent" if any set of demands that can be met on one network can be met on the other network, and vice-versa. They also introduced the notion of bounding networks in that work. One network is said to "bound" another if any set of demands that can be met on the first network can also be met on the second. (In this case the first network lower bounds the second, or, equivalently, the second upper bounds the first.)

The key advantage of this approach is that it does not explicitly find the capacity region of a given network, but relates it to the capacity region of another distinct network whose capacity can be easier to derive. This method is especially interesting when one of the networks is comprised entirely of noiseless bit pipes, since it abstracts the stochastic relationship between the signals transmitted by the nodes of the network. As a result, only the inherent combinatorial network coding problem must be solved in order to describe the set of achievable rates.

The first network equivalence result, introduced in [3], can be stated as follows: If a network is composed entirely of memoryless, noisy, point-to-point links, the set of achievable demands remains unaltered if each noisy link is substituted by a noiseless bit pipe with throughput equal to the capacity of the corresponding noisy channel. Unfortunately, such a result does not hold for networks that contain multi-terminal channels, which comprise most of the cases of interest in wireless systems. In [4], [5], Koetter *et al.* address this issue by investigating upper and lower bounding bit-pipe models for multi-terminal channels. Here, an upper (lower) bounding bit-pipe model for a channel is a collection of bit pipes such that replacing the channel by its bit-pipe model in *any* network that contains it yields a new network whose capacity region is a super-(sub-) set of the capacity region of the original network. Again, this reduces the challenge of bounding the capacity region to its combinatorial core.

In this paper, we introduce a new approach to designing bit-pipe bounding models. The "one-shot" emulation argument developed in this work simplifies the process of deriving upper bounding models and thereby simplifies the derivation of network capacity bounds. Unlike some of the prior approaches (e.g., [1], [2]), it can be applied to any network composed of independent, memoryless channels under any family of demands.

The use of bounding models provides a systematic and straightforward method to bound the capacity region of a general network. Examining network codes for the bounding network provides insight on how to perform network coding on the original network. The results are generally tighter than cut-set bounds and this approach applies more broadly than generalizations of the cut-set approach (e.g., [6]).

The rest of the paper is organized as follows. Section II presents the problem setup. In Section III, a constraint on the total sum rate through an upper bounding bit-pipe model is introduced. This constraint is a necessary condition for the model to be a valid upper bound for a discrete memoryless multi-terminal channel within an arbitrary network. In Section IV, the "one-shot" emulation argument is used to develop a family of bit pipe models that can replace the multiple access channel and a class of broadcast channels within a general network in a bounding sense. Finally, Section V provides a numerical example on how to apply the developed bit-pipe models to a specific network. Concluding remarks are presented in Section VI.

II. Problem Setup

We consider the setup introduced in [3], [5] and follow the notation of [7]. A discrete-time network $\mathcal{N}$ is composed of m nodes, denoted by the set $\mathcal{V} = \{1, ..., m\}$. Each node i is associated with random variables $X^{(i)} \in \mathcal{X}^{(i)}$ and $Y^{(i)} \in \mathcal{Y}^{(i)}$, corresponding to channel inputs and received symbols, respectively. We assume that the network is memoryless and is characterized by the conditional transition probability:

$$p(\mathbf{y}|\mathbf{x}) = p(y^{(1)}, \ldots, y^{(m)}|x^{(1)}, \ldots, x^{(m)}) \ .$$

In order to simplify notation, we assume that each node is connected to only one input channel and one output channel. All of the results presented here immediately extend to the more general case. Each node $v \in \mathcal{V}$ uses a code of block length n with the goal of communicating for each U contained in the power set of $\mathcal{V}\backslash\{v\}$, here denoted by $\mathcal{P}(\mathcal{V}\backslash\{v\})$, the message $W^{\{v\}\to U} \in \mathcal{W}^{\{v\}\to U} \triangleq \{1, \ldots, 2^{nR^{(\{v\}\to U)}}\}$. The messages are independent and uniformly distributed by assumption. A vector of multicast rates $\mathcal{R} \triangleq \{R^{(\{v\}\to U)} : (v, U) \subseteq \mathcal{V} \times \mathcal{P}(\mathcal{V}\backslash\{v\})\}$ is said to be achievable if *all* the messages can be decoded with arbitrarily small error probability. For more details on the setup, we refer the reader to [3]–[5].

The network $\mathcal{N}$ contains an independent channel $\mathcal{C}_i$ denoted by

$$\mathcal{C}_i = \left(\prod_{v \in \mathcal{V}_{i,1}} \mathcal{X}^{(v)}, p_i(\mathbf{y}_i|\mathbf{x}_i), \prod_{v \in \mathcal{V}_{i,2}} \mathcal{Y}^{(v)} \right),$$

where $\mathbf{y}_i = (y^{(j)} : j \in \mathcal{V}_{i,2})$ and $\mathbf{x}_i = (x^{(j)} : j \in \mathcal{V}_{i,1})$, $\mathcal{V}_{i,1}, \mathcal{V}_{i,2} \subseteq \mathcal{V}$, $\mathcal{V}_{i,1} \cap \mathcal{V}_{i,2} = \phi$ if

$$\mathcal{N} = \left(\prod_{v \in \mathcal{V}} \mathcal{X}^{(v)}, p_0(\mathbf{y}_0|\mathbf{x}_0) p_i(\mathbf{y}_i|\mathbf{x}_i), \prod_{v \in \mathcal{V}} \mathcal{Y}^{(v)} \right),$$

where $\mathbf{y}_0 = (y^{(i)} : i \notin \mathcal{V}_{i,2})$, $\mathbf{x}_0 = (x^{(i)} : i \notin \mathcal{V}_{i,1})$. In other words, the channel $\mathcal{C}_i$ can be "decoupled" from the rest of the network. As in [3], we represent the stochastic behavior of the rest of the network by $\mathcal{C}_0$, and denote $\mathcal{N} = \mathcal{C}_0 \times \mathcal{C}_i$.

Let $\mathscr{R}(\mathcal{N}) = \mathscr{R}(\mathcal{C}_0 \times \mathcal{C}_1)$ denote the closure of the set of all rate vectors achievable by the network $\mathcal{N}$. Let $\mathcal{C}_2$ be a discrete, memoryless, multi-terminal channel with the same number of inputs and outputs as $\mathcal{C}_1$. We say that $\mathcal{C}_2$ lower bounds $\mathcal{C}_1$, or, equivalently, $\mathcal{C}_1$ upper bounds $\mathcal{C}_2$, if $\mathscr{R}(\mathcal{C}_0 \times \mathcal{C}_2) \subseteq \mathscr{R}(\mathcal{C}_0 \times \mathcal{C}_1)$. In the following sections, we will refer to upper and lower bounding structures composed entirely of noiseless bit pipes[1] as $\mathcal{S}_u$ and $\mathcal{S}_l$, respectively, and denote noisy multi-terminal channels as $\mathcal{C}_i$ for some subscript i. We represent the bounding relation by $\mathcal{S}_l \subseteq \mathcal{C}_i \subseteq \mathcal{S}_u$.

III. Bounding models composed of bit-pipes

Any point in the capacity region of an isolated multi-terminal channel with independent inputs (i.e., a multi-terminal channel not within a larger network) can serve as a lower bound S_l since operating a channel code across this channel within a network makes it behave essentially as a lossless link [4]. The following theorem shows that any upper bounding bit-pipe model for a multi-terminal channel has a total sum rate at least as large as the total achievable rate over the channel when the receivers and transmitters are allowed to cooperate.

We note that this does not define an equivalence between the models. However, this bound is sharp in the sense that a network exists that contains the multi-terminal channel and can achieve the total sum rate delimited by the upper bound.

[1]We make the common assumption that a bit-pipe might have a non-integer capacity [4].

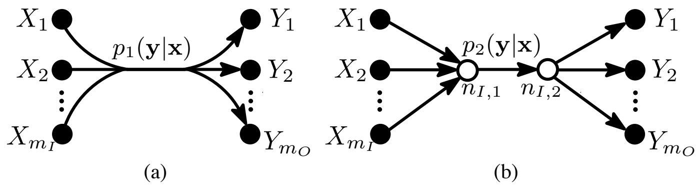

Fig. 1. (a) General multi-terminal channel. (b) Corresponding point-to-point equivalent model.

Theorem 1. *Let $\mathcal{C}_1 = \left(\prod_{i=1}^{m_I} \mathcal{X}^{(i)}, p_1(\mathbf{y}|\mathbf{x}), \prod_{i=1}^{m_O} \mathcal{Y}^{(i)}\right)$ be a multi-terminal channel with m_I inputs and m_O outputs. In addition, let $\mathcal{S}_u$ be a network comprised of noiseless point-to-point bit pipes with m_I inputs and m_O outputs. If $\mathcal{C}_1 \subseteq \mathcal{S}_u$, then the total sum rate achievable through $\mathcal{S}_u$, denoted by $C_{\mathcal{S}_u}$, satisfies:*

$$C_{\mathcal{S}_u} \geq \max_{p(x_1, \ldots, x_{m_I})} I(X_1, \ldots, X_{m_I}; Y_1, \ldots, Y_{m_O}) \quad (1)$$

Remark. It might be tempting to try to use cut-set bounds [7, Theorem 15.10.1] to prove the previous theorem, following a similar approach to the one presented in [8]. However, since it is unclear from cut-set bounds alone which general sets of demands are and are not achievable for an arbitrary network, this would guarantee that the cut-set outer bound of the region remains unaffected but not guarantee that the achievability region of the bounding model includes the one of the original network. For that, we need to use a simple emulation argument and resort to the point-to-point equivalence theorem introduced in [3], as shown below.

Proof: Consider a noisy bounding model $\mathcal{C}_2$ composed of a network with m_I input nodes, two intermediate nodes $n_{I,1}$ and $n_{I,2}$, and m_O output nodes, depicted in Fig. 1. The input nodes are connected to $n_{I,1}$ by noiseless links, each with capacity $\log|\mathcal{X}^{(i)}|$, $i = 1, \ldots, m_I$. In turn, the output nodes are connected to $n_{I,2}$ by noiseless links with capacity $\log|\mathcal{Y}^{(i)}|$, $i = 1, \ldots, m_O$. The intermediate nodes are connected by a discrete memoryless channel with a vector input $[X_1, \ldots, X_{m_I}]$ and a vector output $[Y_1, \ldots, Y_{m_O}]$, where $x_i \in \mathcal{X}_i$, $y_j \in \mathcal{Y}_j$. In addition, this channel has the same transition probability as $\mathcal{C}_1$ but the inputs and outputs are here considered as vectors, i.e.,

$$p_2([y_1, \ldots, y_{m_O}]|[x_1, \ldots, x_{m_I}]) = p_1(\mathbf{y}|\mathbf{x}) \ .$$

Clearly, any coding strategy used in the original channel can be emulated by the input and output nodes of $\mathcal{C}_2$ by imposing that the intermediate nodes simply forward their corresponding inputs. Therefore, $\mathcal{C}_1 \subseteq \mathcal{C}_2$ for any network $\mathcal{N}$. In addition, from the point-to-point equivalence result introduced in [3], the link between the intermediate nodes can be substituted by a noiseless bit pipe with capacity $C_{\mathcal{S}_u}$. The resulting bit-pipe model, denoted by $\mathcal{S}_u$, then also satisfies $\mathcal{C}_1 \subseteq \mathcal{S}_u$.

In addition, a network always exists that can achieve the total maximum sum rate $C_{\mathcal{S}_u}$ through any multi-terminal channel. This can be done by a network which allows all transmitting nodes to cooperate for encoding and all receiving

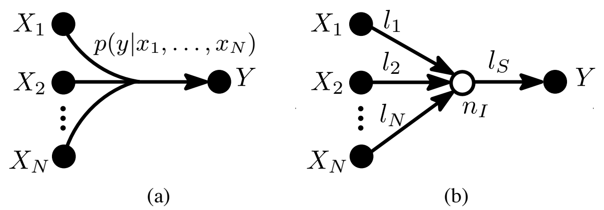

Fig. 2. (a) Discrete memoryless N transmitter multiple access channel. (b) Corresponding bounding model for the N transmitter MAC.

nodes to cooperate for decoding. Therefore, any noiseless bit-pipe upper bounding model for $\mathcal{C}_1$ has to be able to achieve a total sum rate at least as large as $C_{\mathcal{S}_u}$. ■

IV. Point-to-point bounds for multiple access channels and for a class of broadcast channels

In the following sections we introduce upper bounding models for the multiple-access and broadcast channels based on a "one-shot" approach, i.e., where the upper bounding network can emulate *exactly* the lower bounding network at each time step. Even though this approach might seem crude at first, it provides good results in different practical cases, as well as an effective method for bounding the capacity region of large, heterogenous networks. This procedure determines general upper bounds that are sharp in the sense described in Theorem 1.

A. Multiple access channels

Let $\mathcal{C}_{MAC}$ be an N user discrete memoryless multiple access channel defined as $\mathcal{C}_{MAC} = (\prod_i^N \mathcal{X}_i, p(y|x_1,\ldots,x_n), \mathcal{Y})$ with channel inputs $X_1, X_2, \ldots, X_N$ and output Y, depicted in Fig. 2(a). We can create a point-to-point upper bounding model for the MAC shown in Fig. 2(b), where $l_1,\ldots,l_N,l_S$ are noiseless bit pipes with capacity R_{l_i} and n_I is an intermediate node. This bounding structure is denoted by $\mathcal{S} = (R_{l_1},\ldots,R_{l_N},R_{l_S})$.

We can bound the total information flow through this channel by considering that the bit pipe represented by l_S is a point-to-point channel with inputs $\mathbf{X} = (X_1,\ldots,X_N)$, output Y and transition probability $p(y|\mathbf{x}) = p(y|x_1,\ldots,x_n)$. As in Theorem 1 the point-to-point capacity of l_S is equivalent to the maximum rate achievable by the MAC with cooperation among the transmitters, given by:

$$C_S = \max_{p(x_1,\ldots,x_N)} I(Y; X_1, \ldots X_N) \ . \tag{2}$$

The resulting bound is $\mathcal{S}_{u,1} = (\log|\mathcal{X}_1|,\ldots,\log|\mathcal{X}_N|, C_S)$. As discussed previously, this bound is sharp.

The previous approach allows us to bound the total information flow through the MAC, but it does not offer insight on how to find bounds for the maximum transmission rates from the individual users. In order to bound $l_1,\ldots,l_N$, we can explore certain properties of the transition probability $p(y|x_1,\ldots,x_n)$. This can be done by modeling $l_1,\ldots,l_N$ as point-to-point noisy channels, with outputs that can be combined by the intermediate node n_I in such a way as to allow the emulation of the transition probability $p(y|x_1,\ldots,x_N)$.

In particular, assume that there exists a sequence of transition probabilities $p_1(z_1|x_1),\ldots,p_N(z_N|x_N)$, where $x_i \in \mathcal{X}_i$ and $z_i \in \mathcal{Z}_i$ for all i, and a function $g(z_1,\ldots,z_N)$ such that $g : \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_N \to \mathcal{Y}$ and

$$p(y|x_1,\ldots,x_N) = \sum_{\substack{z_1,\ldots,z_N: \\ g(z_1,\ldots,z_N)=y}} \prod_{i=1}^N p(z_i|x_i). \tag{3}$$

At each discrete time step, we can emulate the exact marginal probability of the original MAC by imposing that the intermediate node forwards the value of $g(z_1,\ldots,z_N)$. Therefore, we can substitute each noisy point-to-point link by a noiseless bit pipe of equal capacity. This results in the upper bounding model $\mathcal{S}_{u,2} = (C_1, C_2, \ldots, C_N, \log|\mathcal{Y}|)$ where

$$C_i = \max_{p(x_i)} I(Z_i; X_i), \ i = 1,\ldots,N. \tag{4}$$

We shall now illustrate these results by presenting bounds for the Gaussian and binary additive multiple access channels.

1) Gaussian MAC: Consider a Gaussian MAC with output $Y = a_1X_1 + a_2X_2 + Z$, where $Z \sim \mathcal{N}(0,N)$, $|a_i^2| < 1$ and power constraints $\mathbb{E}[X_i^2] < P_i$ for $i = 1,2$. Using the graph in Fig. 2(b) and the previous approach, we can model l_1 and l_2 by two independent (noisy) Gaussian channels, with noise variances αN and $(1-\alpha)N$, respectively.

The value of α can be chosen in order to minimize the total sum rate through l_1 and l_2, i.e.

$$\alpha^* = \text{argmin}\left[\left(1 + \frac{\text{SNR}_1}{\alpha}\right)\left(1 + \frac{\text{SNR}_2}{(1-\alpha)}\right)\right], \tag{5}$$

giving

$$\alpha^* = \left(1 + \sqrt{\frac{\text{SNR}_1^{-1}+1}{\text{SNR}_2^{-1}+1}}\right)^{-1}, \tag{6}$$

where $\text{SNR}_i = |a_i|^2 P_i/N$. In addition, the total sum rate can be bounded by the maximum cooperation rate as in (2), i.e. $C_{S,MAC} = \gamma((\sqrt{\text{SNR}_1} + \sqrt{\text{SNR}_2})^2)$, where $\gamma(x) = \frac{1}{2}\log(1+x)$. The resulting upper bounding models for this case are $S_{u,1,MAC} = (\infty, \infty, C_{S,MAC})$ and

$$S_{u,2,MAC} = \left(\gamma\left(\frac{\text{SNR}_1}{\alpha^*}\right),\ \gamma\left(\frac{\text{SNR}_2}{1-\alpha^*}\right),\right.$$
$$\left.\gamma\left(\frac{\text{SNR}_1}{\alpha^*}\right) + \gamma\left(\frac{\text{SNR}_2}{1-\alpha^*}\right)\right). \tag{7}$$

The lower bounding bit-pipe model is given by the capacity region for the independent MAC, which is simply:

$$S_{l,MAC} = (\gamma(\text{SNR}_1), \gamma(\text{SNR}_2), \gamma(\text{SNR}_1 + \text{SNR}_2)). \tag{8}$$

The gap between the maximum sum rate of the upper and lower bounding models, measured in bits per channel use, is:

$$\Delta_{\text{MAC}} = \frac{1}{2}\log\left(1 + \frac{2\sqrt{\text{SNR}_1\text{SNR}_2}}{1 + \text{SNR}_1 + \text{SNR}_2}\right) \leq \frac{1}{2} \ . \tag{9}$$

This difference is much smaller at low SNR. Note that, for the channel in isolation, it follows directly that feedback and user cooperation can increase the maximum sum-rate through the two user Gaussian MAC channel in *at most* 1/2 bit, and this gain is much smaller at low SNR. We highlight that this well known gap [9] was proved *directly* from network equivalence results and a simple emulation argument, without the need of any additional proofs.

2) Binary MAC: Consider a binary additive MAC channel, with binary inputs X_1, X_2 and output $Y = X_1 \oplus X_2 \oplus Z$, where Z is a Bernoulli random variable with parameter ϵ and $\oplus$ denotes the XOR operation. Going over the same steps taken for the Gaussian MAC and using the same upper bounding structure, the sum rate of the links from the transmitting nodes to the intermediate node can be modeled as binary symmetric channels with parameters ϵ_1 and ϵ_2, with the given constraint that $\epsilon_1 * \epsilon_2 = \epsilon$. The sum capacity $C_1 + C_2 = 2 - H(\epsilon_1) - H(\epsilon_2)$ of the branches is minimized at $\epsilon_1 = \epsilon_2 = \epsilon^* = \frac{1}{2}(1 - \sqrt{1-2\epsilon})$.

The sum rate, in turn, can be bounded by $1 - H(\epsilon)$. It follows that the corresponding upper bounding bit-pipe structures are $S_{u,1,MAC} = (1, 1, 1 - H(\epsilon))$ and $S_{u,2,MAC} = (1 - H(\epsilon^*), 1 - H(\epsilon^*), 2)$.

The lower bounding sum rate restriction for this channel is also $1 - H(\epsilon)$, and, consequently, cooperation and feedback cannot increase the total rate achieved by a binary symmetric MAC. This, of course, is expected, since the capacity of the considered MAC is limited by a binary symmetric channel, which cannot achieve a throughput larger than $1 - H(\epsilon)$ regardless of the input strategy.

B. Broadcast channels with independent noise

We now turn our attention to a class of two-user discrete memoryless broadcast channels $\mathcal{C}_{BC} = (\mathcal{X}, p(y_1, y_2|x), \mathcal{Y}_1 \times \mathcal{Y}_2)$ displayed in Fig. 3(a). We assume that the transition probability can be factored as $p(y_1, y_2|x) = p(y_1|x)p(y_2|x)$. Consider the "equivalent" model presented in Fig. 3(b), composed by an intermediate node n_I and three links, l_S, l_1 and l_2, with capacities denoted by $\mathcal{S} = (R_{l_S}, R_{l_1}, R_{l_2})$. A lower bounding model $\mathcal{S}_L$ can be found by setting $R_{l_S} = \log|\mathcal{X}|$, and choosing R_{l_1} and R_{l_2} as a known operating point of the broadcast channel. For example, R_{l_1} and R_{l_2} can be chosen as an operating point that maximizes the total sum rate for the channel in isolation.

Analogously to the procedure taken for the MAC channel, an upper bound $\mathcal{S}_{u,1}$ can be found by defining B_1 and B_2 as the maximum achievable individual rates when the broadcast channel is operated as two point-to-point channels. We denote this upper bound as $\mathcal{S}_{u,1} = (\log|\mathcal{X}|, B_1, B_2)$, where B_i, $i = 1, 2$, is the capacity of the point-to-point channel $\mathcal{C}_{y_i} = (\mathcal{X}, p(y_i|x), \mathcal{Y}_i)$ and $p(y_i|x)$ is the corresponding marginal transition probability of $p(y_1, y_2|x)$.

Another upper bounding model is given by modeling l_1 as a two-look point-to-point channel, defined as $\mathcal{C}_{\mathbf{y}} = (\mathcal{X}, p_{\mathbf{y}}(\mathbf{y}|x), \mathcal{Y})$, where $\mathbf{y} = (y_1, y_2)$, $\mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2$ and $p_{\mathbf{y}}((y_1, y_2)|x) = p(y_1, y_2|x)$. Denoting the capacity of $\mathcal{C}_{\mathbf{y}}$ by $B_{\mathbf{y}}$, the corresponding bound is given by $\mathcal{S}_{u,2} = (B_{\mathbf{y}}, \log|\mathcal{Y}_1|, \log|\mathcal{Y}_2|)$ or, equivalently, $\mathcal{S}_{u,2} = (B_{\mathbf{y}}, B_{\mathbf{y}}, B_{\mathbf{y}})$. We shall illustrate these results with two examples.

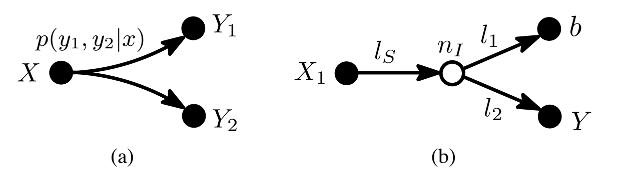

Fig. 3. (a) Two user discrete memoryless broadcast channel. (b) Bounding model for the two user BC.

1) Gaussian broadcast channels with independent noise: Consider the Gaussian broadcast channel $Y_1 = a_1X + Z_1$, $Y_2 = a_2X + Z_2$ and $Z_i \sim \mathcal{N}(0, N_i)$, $i = 1, 2$, $Z_1 \perp\!\!\!\perp Z_2$ and $\mathbb{E}[X^2] \leq P$. It is straightforward to show that the two upper bounding bit-pipe models in this case are

$$\mathcal{S}_{u,BC,1} = (\gamma(\text{SNR}_1) + (\text{SNR}_2), \gamma(\text{SNR}_1), \gamma(\text{SNR}_2)) \tag{10}$$

and, given $C_{S,BC} = \gamma(\text{SNR}_1 + \text{SNR}_2)$, $\mathcal{S}_{u,BC,2} = (C_{S,BC}, C_{S,BC}, C_{S,BC})$ where $\text{SNR}_i = |a_i|^2 P/N_i$, $i = 1, 2$. Considering that $\text{SNR}_2 < \text{SNR}_1$, for any $0 \leq \beta \leq 1$ a lower bound for the channel is the maximum rate achievable by using superposition coding and decoding independently at the receivers:

$$\mathcal{S}_{l,BC} = \left(\infty, \gamma(\beta\text{SNR}_1), \gamma\left(\frac{(1-\beta)\text{SNR}_2}{\beta\text{SNR}_2 + 1}\right)\right). \tag{11}$$

The gap between the total sum rate achievable by the upper bound $\mathcal{S}_{u,BC,2}$ and the lower bounding model, measured in bits per channel use, is:

$$\Delta_{\text{BC}} = \frac{1}{2}\log\left(1 + \frac{\text{SNR}_2}{\text{SNR}_1 + 1}\right) \leq \frac{1}{2}. \tag{12}$$

It also follows that, for the channel in isolation, feedback and cooperation among the receivers can increase capacity in at most 1/2 bit/channel use, and at considerably smaller values at low SNR. Once again, this known result (e.g. [10]) was obtained directly from the point-to-point network equivalence theorem and some additional analysis.

2) Binary symmetric broadcast channels with independent noise: A binary symmetric broadcast channel is defined as $Y_1 = X \oplus Z_1$ and $Y_2 = X \oplus Z_2$, where X, Z_1 and Z_2 are binary and Z_1 and Z_2 are independent Bernoulli random variables with parameters p_1 and p_2, respectively, $0 < p_1 < p_2 < 1$. The upper bounding structure can be easily calculated as $\mathcal{S}_{u,BC,1} = (R_{BSS,1} + R_{BSS,2}, R_{BSS,1}, R_{BSS,2})$ and $\mathcal{S}_{u,BC,2} = (R_{BSS,\mathbf{y}}, R_{BSS,\mathbf{y}}, R_{BSS,\mathbf{y}})$, where

$$\begin{aligned} R_{BSS,\mathbf{y}} &= 1 + H(p_1 * p_2) - H(p_1) - H(p_2) \\ R_{BSS,1} &= 1 - H(p_1) \\ R_{BSS,2} &= 1 - H(p_2) \end{aligned} \tag{13}$$

and $p_1 * p_2 = p_1(1 - p_2) + p_2(1 - p_1)$. The lower bounding model for this scheme is $\mathcal{S}_{l,BC} = (1, H(\beta * p_1) - H(p_1), 1 - H(\beta * p_2))$ for some $0 \leq \beta \leq 1$. The maximum gap between the total sum rate through the lower bower and upper bounding model in bits/channel use is:

$$\Delta_{\text{BC}} = H(p_1 * p_2) - H(p_2) \leq 0.215. \tag{14}$$

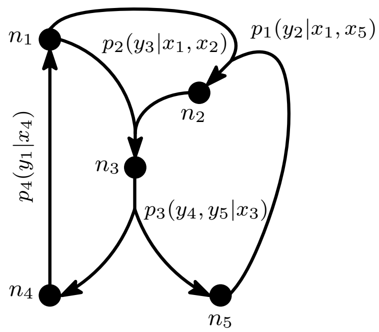

Fig. 4. Example network.

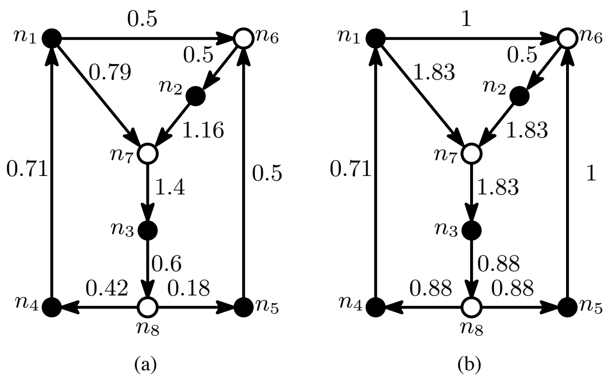

Fig. 5. (a) Lower bounding network. (b) Upper bounding network. The white nodes represent added intermediate nodes. Note that in (a) nodes n_3 and n_8 could be collapsed together.

The value of Δ_{BC} decreases sharply for values of p_1 and p_2 near 0, which indicates that, for channels with small crossover probabilities, feedback and cooperation among decoders does not significantly increase the achievable sum rate.

V. Numerical Example

In order to illustrate the procedure of bounding a network's achievable rate region using the derived models, we now turn our attention to the network depicted in Fig. 4. This network is composed by nodes n_i, $i = 1, \ldots, 5$, two multiple access channels, one broadcast channel and one point-to-point link. The input/output of node n_i at a given instant is represented as (x_i, y_i), and the corresponding transition probabilities are indicated in the figure and are independent among themselves. This network can represent, for example, a heterogenous wireless sensor network.

The transition probability $p_1(y_2|x_1, x_5)$ corresponds to a binary MAC, as described in Section IV-A, where the additive noise is Bernoulli with parameter $\epsilon_1 = 0.11$. Furthermore, let the transition probability $p_2(y_3|x_1, x_5)$ represent a Gaussian MAC where the average received SNR from nodes n_1 and n_2, defined in Section IV-A and denoted by SNR_1 and SNR_2, respectively, are $\mathrm{SNR}_1 = 2$ and $\mathrm{SNR}_2 = 4$.

In addition, let the broadcast channel with transition probability $p_3(y_4, y_5|x_3)$ also be Gaussian, where the average received SNR by nodes n_4 and n_5, as defined in Section IV-B, is given by $\mathrm{SNR}_4 = 1.6$ and $\mathrm{SNR}_5 = 0.8$, respectively. Finally, let the point-to-point channel with transition probability $p_4(y_1|x_4)$ also be a binary symmetric channel with crossover probability $\epsilon_4 = 0.05$.

It is difficult to analyze the capacity of this network using standard methods such as cut-set bounds without laborious computations. Using the family of bounding structures derived in Section IV, we can immediately find the lower and upper bounding networks depicted in Fig. 5(a) and 5(b). All the values are given in terms of bits/channel use. Nodes n_6, n_7 and n_8, marked in white, represent added intermediate nodes for each bounding model.

Note that each point-to-point lower bounding bit pipe leaving an added intermediate node is less than 1/2 bit away from the corresponding upper bounding bit pipe. In addition, this method clearly illustrates the possible losses of decoding and forwarding at each node instead of using a cooperative coding/decoding scheme and the available feedback. This example highlights the key advantage of the network equivalence approach: substituting the channels of a network by their corresponding bounding components is a powerful yet simple method for capacity analysis. The benefits of this approach become even greater for larger networks.

VI. Conclusion

We used network equivalence results to create a systematic procedure for bounding the capacity of large networks that include multi-terminal channels. This method consists of substituting each multi-terminal channel by an appropriate model composed by bit pipes. We have derived a family of sharp bounds for the MAC and a class of broadcast channels, and illustrated how they can be applied through a numerical example. The presented method can be easily extended in order to encompass other channel models.

References

[1] A. S. Avestimehr, S. N. Diggavi, and D. N. C. Tse, "Wireless network information flow: A deterministic approach," *IEEE Trans. Inf. Theory*, vol. 57, no. 4, pp. 1872–1905, 2011.

[2] R. Etkin, D. Tse, and H. Wang, "Gaussian interference channel capacity to within one bit," *IEEE Trans. Inf. Theory*, vol. 54, no. 12, pp. 5534–5562, 2008.

[3] R. Koetter, M. Effros, and M. Médard, "A Theory of Network Equivalence— Part I: Point-to-Point Channels," *IEEE Trans. Inf. Theory*, vol. 57, no. 2, pp. 972–995, 2011.

[4] ——, "A theory of network equivalence, parts I and II," *1007.1033*, Jul. 2010. [Online]. Available: http://arxiv.org/abs/1007.1033

[5] ——, "Beyond network equivalence," in *Proc. 47th Annual Allerton Conference on Communication, Control, and Computing, 2009*, 2009, pp. 997–1004.

[6] G. Kramer and S. Savari, "Capacity bounds for relay networks," in *Proc. 2006 Workshop on Information Theory and its Application*. Citeseer, 2005.

[7] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. Wiley-Interscience, Jul. 2006.

[8] S. Borade, "Network information flow: limits and achievability," in *Proc. 2002 IEEE Int. Symp. on Information Theory.*, 2002, p. 139.

[9] T. Cover and C. Leung, "An achievable rate region for the multiple-access channel with feedback," *IEEE Trans. Inf. Theory*, vol. 27, no. 3, pp. 292–298, 1981.

[10] A. Gamal, "The capacity of the physically degraded gaussian broadcast channel with feedback (Corresp.)," *IEEE Trans. Inf. Theory*, vol. 27, no. 4, pp. 508–511, 1981.